Loop Calculus in Information Theory and Statistical Physics

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Informal Statistical Physics Seminar, U of Maryland

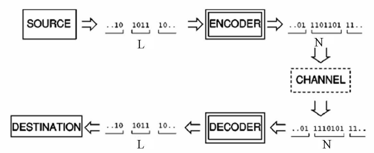
Thanks to M. Stepanov (UofA, Tucson)

Outline

Error Correction

Cell phone Hard disk Optical disk Fiber

Scheme:



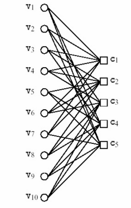
Example of Additive White Gaussian Channel:

$$P(\mathbf{x}_{out}|\mathbf{x}_{in}) = \prod_{i=bits} p(x_{out;i}|x_{in;i})$$

$$p(x|y) \sim \exp(-s^2(x-y)^2/2)$$

- Channel
  is noisy "black box" with only statistical information available
- Encoding:
  use redundancy to redistribute damaging effect of the noise
- Decoding:
  reconstruct most probable codeword by noisy (polluted) channel

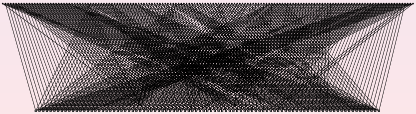
Low Density Parity Check Codes

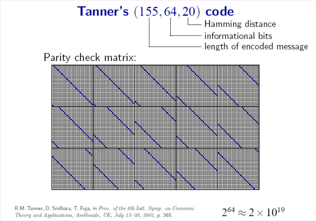


- N bits, M checks, $L = N - M$ information bits
  example: $N = 10$, $M = 5$, $L = 5$
- 2^L codewords of 2^N possible patterns
- Parity check: $\hat{H}\mathbf{v} = \mathbf{c} = \mathbf{0}$
  example:

$$\hat{H} = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \end{pmatrix}$$

- LDPC = graph (parity check matrix) is sparse





Statistical Inference

$\boldsymbol{\sigma}_{\text{orig}}$	$\Rightarrow$	$\mathbf{x}$	$\Rightarrow$	$\boldsymbol{\sigma}$
original *data* $\boldsymbol{\sigma}_{\text{orig}} \in \mathcal{C}$ codeword	noisy channel $\mathcal{P}(\mathbf{x}\|\boldsymbol{\sigma})$	corrupted data: log-likelihood magnetic field	statistical inference	possible preimage $\boldsymbol{\sigma} \in \mathcal{C}$

Maximum Likelihood — symbol Maximum-a-Posteriori

$$\mathrm{ML} = \arg\max_{\boldsymbol{\sigma}} \mathcal{P}(\mathbf{x}|\boldsymbol{\sigma}) \qquad \mathrm{MAP}_i = \arg\max_{\sigma_i} \sum_{\boldsymbol{\sigma}\setminus\sigma_i} \mathcal{P}(\mathbf{x}|\boldsymbol{\sigma})$$

Exhaustive search is generally expensive: complexity $\sim 2^N$

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$$\boldsymbol{\sigma} = (\sigma_1, \cdots, \sigma_N), \quad N \ \textit{finite}, \quad \sigma_i = \pm 1 \ (\textit{example})$$

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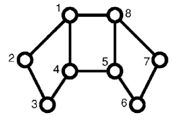
Graphical models of Statistical Inference

Factorization (Forney '01, Loeliger '01)

$$\mathcal{P}(\boldsymbol{\sigma}|\mathbf{x}) = Z^{-1}\prod_a f_a(\mathbf{x}_a|\boldsymbol{\sigma}_a)$$

$$\underbrace{Z(\mathbf{x}) = \sum_{\boldsymbol{\sigma}}\prod_a f_a(\mathbf{x}_a|\boldsymbol{\sigma}_a))}_{\text{partition function}}$$

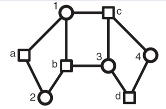


$$f_a \geq 0$$
$$\sigma_{ab} = \sigma_{ba} = \pm 1$$
$$\boldsymbol{\sigma}_1 = (\sigma_{12}, \sigma_{14}, \sigma_{18})$$
$$\boldsymbol{\sigma}_2 = (\sigma_{12}, \sigma_{13})$$

Example: Error-Correction (linear code, bipartite Tanner graph)

$$f_i(h_i|\boldsymbol{\sigma}_i) = \exp(\sigma_i h_i)\cdot\begin{cases} 1, & \forall \alpha,\beta \ni i, \quad \sigma_{i\alpha} = \sigma_{i\beta} \\ 0, & \text{otherwise} \end{cases}$$

$$f_\alpha(\boldsymbol{\sigma}_\alpha) = \delta\left(\prod_{i\in\alpha}\sigma_i, +1\right)$$



h_i - log-likelihoods

Variational Method in Statistical Mechanics

$$P(\boldsymbol{\sigma}) = \frac{\prod_a f_a(\boldsymbol{\sigma}_a)}{Z}, \quad Z \equiv \sum_{\boldsymbol{\sigma}} \prod_a f_a(\boldsymbol{\sigma}_a)$$

Exact Variational Principle — Kullback-Leibler '51

$$F\{b(\boldsymbol{\sigma})\} = -\sum_{\boldsymbol{\sigma}} b(\boldsymbol{\sigma}) \sum_a f_a(\boldsymbol{\sigma}_a) - \sum_{\boldsymbol{\sigma}} b(\boldsymbol{\sigma}) \ln b(\boldsymbol{\sigma})$$

$$\left.\frac{\delta F}{\delta b(\boldsymbol{\sigma})}\right|_{b(\boldsymbol{\sigma})=p(\boldsymbol{\sigma})} = 0 \quad \text{under} \quad \sum_{\boldsymbol{\sigma}} b(\boldsymbol{\sigma}) = 1$$

Variational Ansatz

- Mean-Field: $p(\boldsymbol{\sigma}) \approx b(\boldsymbol{\sigma}) = \prod_i b_i(\sigma_i)$
- Belief Propagation:

$$p(\boldsymbol{\sigma}) \approx b(\boldsymbol{\sigma}) = \frac{\prod_a b_a(\boldsymbol{\sigma}_a)}{\prod_{(a,b)} b_{ab}(\sigma_{ab})} \quad \text{(exact on a tree)}$$

$$b_a(\boldsymbol{\sigma}_a) = \sum_{\boldsymbol{\sigma}\setminus\boldsymbol{\sigma}_a} b(\boldsymbol{\sigma}), \quad b_{ab}(\sigma_{ab}) = \sum_{\boldsymbol{\sigma}\setminus\sigma_{ab}} b(\boldsymbol{\sigma})$$

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Bethe free energy: variational approach
(Yedidia,Freeman,Weiss '01 - inspired by Bethe '35, Peierls '36)

$$F = \underbrace{-\sum_a \sum_{\boldsymbol{\sigma}_a} b_a(\boldsymbol{\sigma}_a) \ln f_a(\boldsymbol{\sigma}_a)}_{\text{self-energy}} + \underbrace{\sum_a \sum_{\boldsymbol{\sigma}_a} b_a(\boldsymbol{\sigma}_a) \ln b_a(\boldsymbol{\sigma}_a) - \sum_{(a,c)} b_{ac}(\sigma_{ac}) \ln b_{ac}(\sigma_{ac})}_{\text{configurational entropy}}$$

$$\forall\ a;\ c \in a: \quad \sum_{\boldsymbol{\sigma}_a} b_a(\boldsymbol{\sigma}_a) = 1, \quad b_{ac}(\sigma_{ac}) = \sum_{\boldsymbol{\sigma}_a \backslash \sigma_{ac}} b_a(\boldsymbol{\sigma}_a)$$

$\Rightarrow$ Belief-Propagation Equations: $\left.\frac{\delta F}{\delta b}\right|_{\text{constr.}} = 0$

MAP≈BP=Belief-Propagation (Bethe-Peirls): iterative ⇒ Gallager '61; MacKay '98

- Exact on a tree ▸ Derivation Sketch
- Trading optimality for reduction in complexity: $\sim 2^L \to \sim L$
- BP = solving equations on the graph:

$$\eta_{\alpha j} = h_j + \sum_{\beta \neq \alpha}^{j \in \beta} \tanh^{-1}\left(\prod_{i \neq j}^{i \in \beta} \tanh \eta_{\beta i}\right) \quad \Leftarrow \text{LDPC representation}$$

- Message Passing = iterative BP
- Convergence of MP to minimum of Bethe Free energy can be enforced

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Linear Programming version of Belief Propagation

In the limit of large SNR, $\ln f_a \to \pm\infty$: BP$\to$LP

Minimize $F \approx E = -\sum_a \sum_{\boldsymbol{\sigma}_a} b_a(\boldsymbol{\sigma}_a) \ln f_a(\boldsymbol{\sigma}_a) =$ self energy

under set of linear constraints

LP decoding of LDPC codes — Feldman, Wainwright, Karger '03

- ML can be restated as an LP over a codeword polytope
- LP decoding is a "local codewords" relaxation of LP-ML
- Codeword convergence certificate
- Discrete and Nice for Analysis
- Large polytope $\{b_\alpha, b_i\} \Rightarrow$ Small polytope $\{b_i\}$

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1. Introduction
   - Main Example: Error Correction
   - Statistical Inference
   - Graphical Models
   - Bethe Free Energy and Belief Propagation (BP)
2. Loop Calculus
   - Gauge Transformations and BP
   - Loop Series in terms of BP
3. Applications
   - Analysis and Improvement of LDPC-BP/LP Decoding
   - Long Correlations and Loops in Statistical Mechanics
4. Conclusions

BP does not account for Loops

Questions:

- Is BP just a heuristic in a loopy case?
- Why does it (often) work so well?
- Does exact inference allow an expression in terms of BP?
- Can one correct BP systematically?

Previous Considerations:

- Rizzo, Montanari '05 - Corrections to BP approximation
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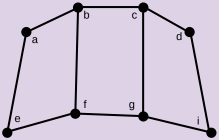
Chertkov,Chernyak '06

Local Gauge, G, Transformations



$$f_a(\boldsymbol{\sigma}_a = (\sigma_{ab}, \cdots)) \to \sum_{\sigma'_{ab}} G_{ab}\left(\sigma_{ab}, \sigma'_{ab}\right) f_a(\sigma'_{ab}, \cdots)$$

$$\sum_{\sigma_{ab}} G_{ab}(\sigma_{ab}, \sigma')G_{ba}(\sigma_{ab}, \sigma'') = \delta(\sigma', \sigma'')$$

The partition function is invariant under any G-gauge!

$$Z = \sum_{\boldsymbol{\sigma}} \prod_a f_a\left(\boldsymbol{\sigma}_a\right) = \underbrace{\sum_{\boldsymbol{\sigma}} \prod_a \left(\sum_{\boldsymbol{\sigma}'_a} f_a(\boldsymbol{\sigma}'_a) \prod_{b \in a} G_{ab}(\sigma_{ab}, \sigma'_{ab}) \right)}_{\text{graphical trace}}$$

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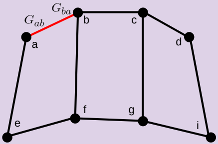

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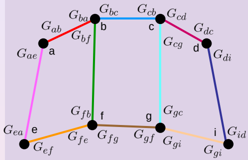

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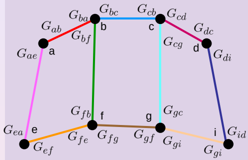

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Gauge Transformation: Binary Representation

$$Z=\sum_{\boldsymbol{\sigma}}\prod_a f_a(\boldsymbol{\sigma}_a)=\sum_{\boldsymbol{\sigma}'}\prod_a f_a(\boldsymbol{\sigma}_a)\prod_{bc}\frac{1+\sigma_{bc}\sigma_{cb}}{2},\quad \sigma_{bc}\neq\sigma_{cb}$$

The binary trick

$$1+\pi\sigma=\frac{\exp(\sigma\eta+\pi\chi)}{\cosh(\eta+\chi)}\left(1+(\tanh(\eta+\chi)-\sigma)(\tanh(\eta+\chi)-\pi)\cosh^2(\eta+\chi)\right)$$

$$\tilde{f}_a(\boldsymbol{\sigma}_a)=f_a(\boldsymbol{\sigma}_a)\prod_{b\in a}\exp(\eta_{ab}\sigma_{ab})$$

$$V_{bc}\left(\sigma_{bc},\sigma_{cb}\right)=1+\left(\tanh(\eta_{bc}+\eta_{cb})-\sigma_{bc}\right)\left(\tanh(\eta_{bc}+\eta_{cb})-\sigma_{cb}\right)\cosh^2(\eta_{bc}+\eta_{cb})$$

Graph Coloring

$$Z=\Big(\prod_{bc}2\cosh(\eta_{bc}+\eta_{cb})\Big)^{-1}\sum_{\boldsymbol{\sigma}'}\prod_a\tilde{f}_a(\boldsymbol{\sigma}_a)\cdot\prod_{bc}V_{bc}$$

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$$Z=\underbrace{Z_0(\eta)}_{\text{ground state}}+\underbrace{\sum_{\text{all possible colorings of the graph}}\cdots}_{\text{excited states}}$$

Gauges and BP

Partition function in the colored representation

$$Z=(\prod_{bc}2\cosh(\eta_{bc}+\eta_{cb}))^{-1}\sum_{\boldsymbol{\sigma}'}\prod_a\tilde{f}_a\prod_{bc}V_{bc},\quad \tilde{f}_a(\boldsymbol{\sigma}_a;\boldsymbol{\eta}_a)=f_a(\boldsymbol{\sigma}_a)\prod_{b\in a}\exp(\eta_{ab}\sigma_{ab})$$

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Fixing the gauges ⇒ BP equations!!

$$\sum_{\boldsymbol{\sigma}_a}\left(\tanh(\eta_{ab}^{(bp)}+\eta_{ba}^{(bp)})-\sigma_{ab}\right)\tilde{f}_a(\boldsymbol{\sigma}_a;\boldsymbol{\eta}_a)=0\quad\Rightarrow\quad\underbrace{\eta_{\alpha j}^{bp}=h_j+\sum_{\beta\neq\alpha}^{j\in\beta}\tanh^{-1}\Big(\prod_{i\neq j}^{i\in\beta}\tanh\eta_{\beta i}^{bp}\Big)}_{\text{LDPC case}}$$

Color Principe: no loose ends

$$\prod_{(bc)}V_{bc}=1+\sum_{\text{colored edges}}*\cdots*\cdots*\cdots$$

Variational Principle:

$$\prod_{(bc)}V_{bc}\to1,\quad Z\to Z_0,\quad \left.\frac{\delta Z_0}{\delta\eta_{ab}}\right|_{\eta^{(bp)}}=0$$

$$Z_0=(\prod_{bc}2\cosh(\eta_{bc}+\eta_{cb}))^{-1}\sum_{\boldsymbol{\sigma}}\prod_a\tilde{f}_a(\boldsymbol{\sigma}_a)$$

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$$\sum_{\boldsymbol{\sigma}_a}\left(\tanh(\eta_{ab}^{(bp)}+\eta_{ba}^{(bp)})-\sigma_{ab}\right)\tilde{f}_a(\boldsymbol{\sigma}_a;\boldsymbol{\eta}_a)=0\quad\Rightarrow\quad\underbrace{\eta_{\alpha j}^{bp}=h_j+\sum_{\beta\neq\alpha}^{j\in\beta}\tanh^{-1}(\prod_{i\neq j}^{i\in\beta}\tanh\eta_{\beta i}^{bp})}_{\text{LDPC case}}$$

Color Principe: no loose ends

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Variational Principe:

$$\prod_{(bc)}V_{bc}\to 1,\quad Z\to Z_0,\quad \left.\frac{\delta Z_0}{\delta\eta_{ab}}\right|_{\eta^{(bp)}}=0$$

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Gauges and BP

Partition function in the colored representation

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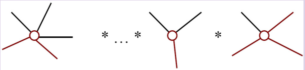

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Loop Series: Chertkov,Chernyak '06

Exact (!!) expression in terms of BP

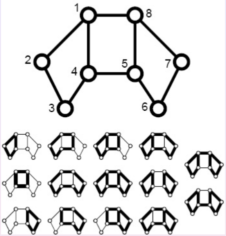
$$Z=\sum_{\boldsymbol{\sigma}_\sigma}\prod_a f_a(\boldsymbol{\sigma}_a)=Z_0\left(1+\sum_C r(C)\right)$$

$$r(C)=\frac{\prod\limits_{a\in C}\mu_a}{\prod\limits_{(ab)\in C}(1-m_{ab}^2)}=\prod_{a\in C}\tilde{\mu}_a$$

$C\in$ Generalized Loops $=$ Loops without loose ends

$$m_{ab}=\int d\boldsymbol{\sigma}_a b_a^{(bp)}(\boldsymbol{\sigma}_a)\sigma_{ab}$$

$$\mu_a=\int d\boldsymbol{\sigma}_a b_a^{(bp)}(\boldsymbol{\sigma}_a)\prod_{b\in a,C}(\sigma_{ab}-m_{ab})$$



- The Loop Series is finite
- All terms in the series are calculated within BP
- BP is exact on a tree
- BP is a Gauge fixing condition. Other choices of Gauges would lead to different representation.

Features of the Loop Calculus

$$Z = Z_0(1 + \sum_C r_C),\ r_C = \prod_{a \in C} \tilde{\mu}_a$$

- Bethe Free Energy is related to the "ground state" term in the partition function: $F(b^*(\eta)) = -\ln Z_0(\eta)$, where $b^*_a(\boldsymbol{\sigma}_a) = \frac{f_a(\boldsymbol{\sigma}_a)\exp(\sum_{b \in a} \eta_{ab}\sigma_{ab})}{\sum_{\boldsymbol{\sigma}_a} f_a(\boldsymbol{\sigma}_a)\exp(\sum_{b \in a} \eta_{ab}\sigma_{ab})}, \quad b^*_{ab}(\sigma_{ab}) = \frac{\exp((\eta_{ab}+\eta_{ba})\sigma_{ab})}{2\cosh(\eta_{ab}+\eta_{ba})}$
- Extrema of $F(b)$ are in one-to-one correspondence with extrema of $Z_0(\eta)$.
- Loop series can be built around any extremum (minimum, maximum or saddle-point) of the Bethe Free energy.
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- Linear Programming limit of the Loop Calculus is well defined.
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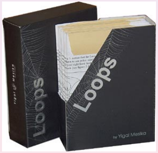
1. Introduction
   - Main Example: Error Correction
   - Statistical Inference
   - Graphical Models
   - Bethe Free Energy and Belief Propagation (BP)

2. Loop Calculus
   - Gauge Transformations and BP
   - Loop Series in terms of BP



3. Applications
   - Analysis and Improvement of LDPC-BP/LP Decoding
   - Long Correlations and Loops in Statistical Mechanics

If BP/LP fails while ML/MAP would not ... one needs to account for Loops

- How many loops are critical to recover from the failure?
- Will accounting for a single most important loop be sufficient?
- How long is the critical loop?
- Will it be difficult to find the critical loop?
- If there are many ... how are the critical loops distributed over scales?

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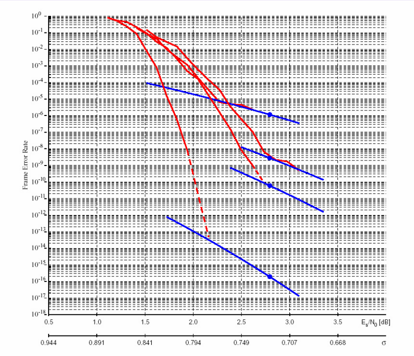
Error-Floor



T. Richardson, Allerton '03

- BER vs SNR = measure of performance
- Waterfall $\leftrightarrow$ Error-floor
- ML and BP/LP are generally different at $s^2 = E_s/N_0 \to \infty$, $FER_{\mathrm{ML}} \sim \exp(-d_{\mathrm{ML}} s^2/2)$ vs $FER_{\mathrm{sub}} \sim \exp(-d_{\mathrm{sub}} s^2/2)$ where $d_{\mathrm{ML}} \geq d_{\mathrm{sub}}$
- Monte-Carlo is useless at FER $\lesssim 10^{-8}$
- Need an efficient method to analyze error-floor

Pseudo-codewords and Instantons

Error-floor is caused by Pseudo-codewords:

Wiberg '96; Forney et.al'99; Frey et.al '01; Richardson '03; Vontobel, Koetter '04-'06

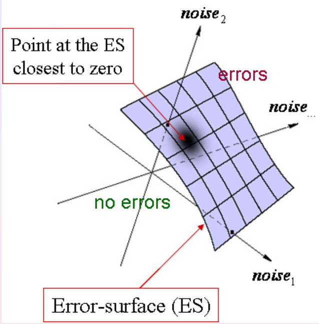
Instanton = optimal conf of the noise

$$BER = \int d(noise)\ WEIGHT(noise)$$

$$BER \sim WEIGHT\left(\begin{array}{c} optimal\ \ conf \\ of\ \ the\ \ noise \end{array}\right)$$

$$\begin{array}{c} optimal\ \ conf \\ of\ \ the\ \ noise \end{array} = \begin{array}{c} Point\ \ at\ \ the\ \ ES \\ closest\ \ to\ \text{"}0\text{"} \end{array}$$

Instantons are decoded to Pseudo-Codewords



Instanton-amoeba

= optimization algorithm

Stepanov, et.al '04,'05

Stepanov, Chertkov '06

Loop Calculus & Pseudo-Codeword Analysis

Single loop truncation

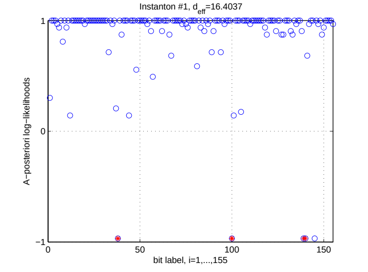
$$Z = Z_0(1 + \sum_C r_C) \approx Z_0(1 + r(\Gamma))$$

Synthesis of Pseudo-Codeword Search Algorithm (Chertkov, Stepanov '06) & Loop Calculus

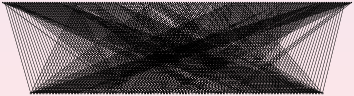
- Consider pseudo-codewords one after other
- For an individual pseudo-codeword/instanton identify a critical loop, Γ, giving major contribution to the loop series.
- Hint: look for single connected loops and use local "triad" contributions as a tester: $r(\Gamma) = \prod_{\alpha \in \Gamma} \tilde{\mu}_\alpha^{(bp)}$

Proof-of-Concept test [(155, 64, 20) code over AWGN]

- $\forall$ pseudo-codewords with $16.4037 < d < 20$ ($\sim$ 200 found) there always exists a simple single-connected critical loop(s) with $r(\Gamma) \sim 1$.
- Pseudo-codewords with the lowest d show $r(\Gamma) = 1$
- Invariant with respect to other choices of the original codeword



Bigger Set



Extended Variational Principe & Loop-Corrected BP

Bare BP Variational Principe:

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New choice of Gauges guided by the knowledge of the critical loop Γ

$$\left.\frac{\delta \exp(-\mathcal{F})}{\delta \eta_{ab}}\right|_{\eta_{\rm eff}} = 0, \quad \mathcal{F} \equiv -\ln(Z_0 + Z_\Gamma)$$

BP-equations are modified along the critical loop Γ

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Loop-Corrected BP Algorithm

- **1.** Run bare BP algorithm. Terminate if BP succeeds (i.e. a valid code word is found).
- **2.** If BP fails find the most relevant loop Γ that corresponds to the maximal $|r_\Gamma|$. Triad search is helping.
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Extended Variational Principe & Loop-Corrected BP

Bare BP Variational Principle:

$$\left.\frac{\delta Z_0}{\delta \eta_{ab}}\right|_{\eta^{(bp)}} = 0, \qquad Z_0 = \left(\prod_{bc} 2\cosh(\eta_{bc}+\eta_{cb})\right)^{-1} \sum_{\boldsymbol{\sigma}} \prod_a P_a(\boldsymbol{\sigma}_a)\Big|_{\eta^{(bp)}}$$

New choice of Gauges guided by the knowledge of the critical loop Γ

$$\left.\frac{\delta \exp(-\mathcal{F})}{\delta \eta_{ab}}\right|_{\eta_{\mathrm{eff}}} = 0, \quad \mathcal{F} \equiv -\ln(Z_0 + Z_\Gamma)$$

BP-equations are modified along the critical loop Γ

$$\left.\frac{\sum_{\boldsymbol{\sigma}_a} (\tanh(\eta_{ab}+\eta_{ba}) - \sigma_{ab}) P_a(\boldsymbol{\sigma}_a)}{\sum_{\boldsymbol{\sigma}_a} P_a(\boldsymbol{\sigma}_a)}\right|_{\eta_{\mathrm{eff}}} = \left.\frac{\prod_{d\in\Gamma} \mu_{d;\Gamma}}{\prod_{(a'b')\in\Gamma}(1-(m^{(*)}_{a'b'})^2)}\, \delta m_{a\to b;\Gamma}\right|_{\eta_{\mathrm{eff}}} \neq 0 \quad [\text{along } \Gamma]$$

Loop-Corrected BP Algorithm

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LP-erasure = simple heuristics

- **1.** Run LP algorithm. Terminate if LP succeeds (i.e. a valid code word is found).
- **2.** If LP fails, find the most relevant loop Γ that corresponds to the maximal amplitude $r(\Gamma)$.
- **3.** Modify the log-likelihoods along the loop Γ introducing a shift towards zero, i.e. introduce a complete or partial erasure of the log-likelihoods at the bits. Run LP with modified log-likelihoods. Terminate if the modified LP succeeds.
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$(155, 64, 20)$ Test

- **IT WORKS!**
  All troublemakers ($\sim$ 200 of them) previously found by LP-based Pseudo-Codeword-Search Algorithm method were successfully corrected by the LP-erasure algorithm.
- Method is invariant with respect the choice of the codeword (used to generate pseudo-codewords).

General Conjecture:

- Loop-erasure algorithm is capable of reducing the error-floor
- Bottleneck is in finding the critical loop
- Local adjustment of the algorithm, anywhere along the critical loop, in the spirit of the Facet Guessing (Dimakis, Wainwright '06), may be sufficient

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Dilute Gas of Loops: $Z = Z_0(1+\sum_C r_C) \approx Z_0 \cdot \prod_{C_{sc}=\text{single connected}} (1+r_{sc})$

Applies to

- Lattice problems in high spatial dimensions
- Large Erdös-Renyi problems (random graphs with controlled connectivity degree)

- The approximation allows an easy multi-scale re-summation
- In the para-magnetic phase and $\mathbf{h}=0$: the only solution of BP is a trivial one $\boldsymbol{\eta}=0$, $Z_0 \to 1$, and the Loop Series is reduced to the high-temperature expansion [Domb, Fisher, et al '58-'90]

Ising model in the factor graph terms

$$Z=\sum_{\boldsymbol{\sigma}}\prod_{\alpha=(i,j)\in X}\exp\left(J_{ij}\sigma_i\sigma_j\right)=\sum_{\boldsymbol{\sigma}}\prod_{a\in\{i\}\cup\{\alpha\}} f_a(\boldsymbol{\sigma}_a)$$

$$f_i(\boldsymbol{\sigma}_i)=\begin{cases}\exp(h_i\sigma_i), & \sigma_{i\alpha}=\sigma_{i\beta}=\sigma_i \quad \forall \alpha,\beta\ni i\\ 0, & \text{otherwise;}\end{cases}$$

$$f_\alpha\left(\boldsymbol{\sigma}_\alpha=(\sigma_{\alpha i},\sigma_{\alpha j})\right)=\exp\left(J_{ij}\sigma_{\alpha i}\sigma_{\alpha j}\right)$$

Loop Series trivially pass the common "loop" tests (from Rizzo, Montanari '05)

- Evaluation of the critical temperature in the constant exchange, zero field Ising model
- Leading $1/N$ corrections to the Free Energy of the Viana-Bray model in the vicinity of the critical point (glass transition)

Results

- BP is better then just a heuristic in the loopy case ... BP is the special Gauge condition eliminating all contributions but loops.
- Exact Marginal probability allows explicit Loop Series expression in terms of a solution of the Belief Propagation equations.
- Truncation and/or Re-summation of the Loop Series provide hierarchy of systematically improvable approximations/algorithms. Standard BP/LP is a first member in the hierarchy.
- Local example (truncation). Finding a critical loop, or a small number of critical loops, can be algorithmically sufficient for drastic improvement of BP decoding in the error-floor domain.
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Future Challenges

- Better Algorithms: Loop Series Truncation/Resummation
- Generalizations. q-ary and continuous alphabets. Quantum spins, Quantum error-correction.
- Loop calculus based analysis of graph ensembles, e.g. understanding and improving the cavity method [Mézard, Parisi '85-'03]
- Extending the list of Loop Calculus Applications, e.g. SAT and cryptography
- Non-BP gauges, e.g. for stat problems on regular and irregular lattices
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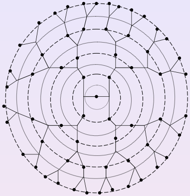
All papers are available at http://cnls.lanl.gov/∼chertkov/pub.htm



$$Z(\mathbf{h}) = \sum_{\boldsymbol{\sigma}} \prod_{\alpha=1}^{M} \delta\left(\prod_{i\in\alpha} \sigma_i, 1\right) \exp\left(\sum_{i=1}^{N} h_i\sigma_i\right)$$

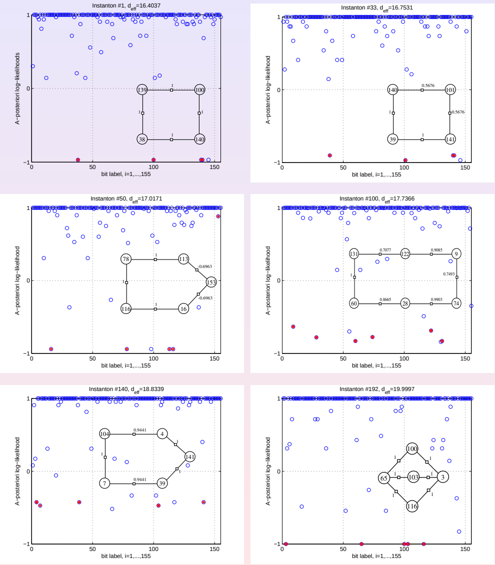
h_i is a log-likelihood at a bit (outcome of the channel)

$$Z^{\pm}_{j\alpha}(\mathbf{h}^{>}) \equiv \sum_{\boldsymbol{\sigma}^{>}}^{\sigma_j=\pm 1} \prod_{\beta>} \delta\left(\prod_{i\in\beta} \sigma_i, 1\right) \exp\left(\sum_{i>} h_i\sigma_i\right)$$

$$Z^{\pm}_{j\alpha} = \exp(\pm h_j) \prod_{\beta\neq\alpha}^{j\in\beta} \frac{1}{2}\left(\prod_{i\neq j}^{i\in\beta}(Z^{+}_{i\beta} + Z^{-}_{i\beta}) \pm \prod_{i\neq j}^{i\in\beta}(Z^{+}_{i\beta} - Z^{-}_{i\beta})\right)$$

$$\eta_{j\alpha} \equiv \frac{1}{2}\ln\left(\frac{Z^{+}_{j\alpha}}{Z^{-}_{j\alpha}}\right), \quad \eta_{j\alpha} = h_j + \sum_{\beta\neq\alpha}^{j\in\beta} \tanh^{-1}\left(\prod_{i\neq j}^{i\in\beta} \tanh \eta_{i\beta}\right)$$

Bethe Free Energy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Instanton #33, d_eff=16.7531
Instanton #50, d_eff=17.0171
Instanton #100, d_eff=17.7366
Instanton #140, d_eff=18.8339
Instanton #192, d_eff=19.9997
A−posteriori log−likelihoods
A−posteriori log−likelihood
bit label, i=1,...,155

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